The Hecke category (part I|factorizable structure)

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! $h_{std\ x}$!Bun_n

In this lecture and the next, we will describe the \Hecke category", namely, the thing which acts on D-modules on Bun_G and with respect to which action the notion of Hecke eigensheaves is de ned. In fact, almost none of this content actually concerns Bun_G, so before we move into talking about something apparently completely di erent, we will give a general description of the goal and indicate why the context must change. Throughout this lecture, our D-modules are assumed to be holonomic.

The Hecke stack; motivation

Back in the very rst lecture, Dennis described some particular examples of Hecke functors for Bun= $Bun_{GL_n}(X)$ (as always, X is the smooth projective curve we are using). They all concerned diagrams

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He also gave other examples of possible \Hecke stacks" with progressively elaborate conditions on , and de n ed corresponding \Hecke functors"

$std_x: D\text{-}mod(Bun_n) - D\text{-}mod(Bun_n),$

$$H^{s}_{t,d}(x_F) = (!\,h^{s}_{t,d}(x), (h^{s}_{t,d}(x))_{F[n-1]})$$

(or with other shifts, for the other stacks). He then indicated that we would need to consid er the category of Hecke functors in order to properly state the eigensheaf condition. Given the above, seemingly ad-hoc descriptio n, it would appear impossible to give a reasonable description of this category. In fact, however, such a description exists and is very natur ally given in terms of the a ne grassmannian Gr_G, which we will review and generalize in this lecture. Recall that for any group

G, the a ne grassmannian is the functor, de n ed on a ne schemes S = Spec R as:

(T;t) (S x) = (

S = C [[z]] and b K = C ((z))

H S

Vi 1 are vector bundles of rank n on X = > !V

Hstd x(S) = = S X : Vs is an injective map of coherent sheaves coker( ) is at over S, supported on fxg ; :

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where x2X(C) and the middle object is a stack

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GrG Tis a G-torsor on the formal disk SpecR[[z]] = Dt:) (1)
T0!Tis a trivialization of Ton Dsnf0gSS= D S

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fraction eld of the completed local ring bOx x, and we will often identify by O 1x

and that the group ind-scheme G(b) acts on it by changing the trivialization. (f0g
K x

notation, then, Grby choosing a uniformizing parameter znear x; when this happens, we will For any group G, not just GL_n
just write b Kand$_G$
!T is an isomorphism on X^2 nfxg$_S$
T are G-torsors on X_S) :
i S
$H_X(S)$ = ((T_1;T_2;) : T_1
There is again
a convolution
diagram
Bun$_{hG}$ $_x$ H$_{hx}$!$_x$!Bun$_G$
G G 1 1on X), the ber of h over , restricted to D
Ton D , then T
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For any point of Bun(C) (that is, a G-torsor T. Indeed, if we
(noncanonically) pick a trivialization of T

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he F=b(O$_x$induce a global strati cation of) Gr$_G$ 1 i is actually a Gr$_G$ S sn 2
H [-bundle over Bun$_G$
G
) acting on Gr x
)!(M e F);
G

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F x(M) = (! h
X$_n$ = H$_n$

x

Here, x

and the ber of h
is
noncanonically identi ed with Gr, can vary over all possible G-torsors and over
all trivializations, since Tis now trivial (this is the Beauville{Laszlo theorem, which
says that we can always glue on Taway from xto complete T). It is not hard to
show (using this same logic) that H, where the structure group is in fact G(); we
will return to this more precisely next time. Therefore, the G(b O)-orbits on Gr$_x$; it
turns out that their various closures are exactly the strange Hecke stacks
considered before.
Recall the de nition of equivariance of a D-module with respect to the action of
a group on the underlying space; in the case of G(b O$_{xG}$, it means that the two
pullbacksG(b O$_{xG}$ $_{apr}$are isomorphic, with the isomorphism subject to some
natural conditions. Any such D-module Fcan be extended along H$_{xx}$) to a

\twisted pullback" e F; for M2D-mod(Bun), set MM e F Hxthis is the uniform de nition of the Hecke functors. We see, therefore, that the Hecke category of Hecke functors is simply the category D-mod$_{G(\hat{O}_x)}$(Gr$_G$) or, as we will call it later, Sph. One further modi cation is possible. If x2X(C) is not xed but allowed to vary, or to multiply toseveral points, then there arise relative and higher Hecke stacks Hde ned by

!T$_2$ is an isomorphism on X x)

$H_n(S)$ = ((~x;T$_1$;T$_2$;) : T$_1$ ~x2X$_n$(S);T$_i$ are G-torsors on X) :

$_i$) is the graph of x_i : S!Xinside X$_S$. There are diagrams

n Bun$_{hG}$ $_n$ H$_{hn}$!$_n$!Bun X$_n$

is something we have not seen yet but which we will introduce presently: the \factorizable" grassmannian.

The factorizable grassmannian

(S) = (: for a xed choice of x2X(C), we have: ^{G} (T;t)

Recall the \global" version of Gr 2

Gr$_G$ Tis a G-torsor on X$_S$tis a trivialization of Ton X$_S$ nfxg

= =

This has the same relationship to x as does H_x, and the dependency problem is solved in the same way by de ning unrestricted and relative versions:

$\mathrm{Gr}_{G;X^n}(S) = \mathrm{Gr}_n(S) =$ $((\vec{x};T;t))$ ^{n} T is a G-torsor on X_S x) : t is a trivialization of T on X_S n [$_i$)

Note that X can be any smooth curve in this definition, not necessarily complete (or, indeed, even algebraic). Th ese are all ind-proper schemes over X, and they have a number of relationships comprising the factorizable structure: be the corresponding

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of
X
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n. Then there are isomorphisms Gr n p be the open subset of X n

1 which are compatible with re nement of the partition

p; Let p be a partition as above and suppose its parts

p;:::;x n) such that if x ii = x j

For n;m2N , let p be a partition of [1;n] into m parts and inside X

j = Gr m

p

have sizes n i; let U

Gr ni U p

consisting of coordinates (x

Gr n

p

, then i;j are in the same part of p. Then there are isomorphisms j U p Y compatible with re nement of t

= U's). Furthermore, these isomorp

some diagonal, and away from o

on

X n

which is compatible with both of the above classes of isom

For any n, an equivariance structure for the action of the symmetric group S

It is possible to give a precise statement of the nature of these c

little reward for the necessary work, it is in the appendix. The pr

= = x n1

n n m+1

j

i j 2p and V i j are the same in both cases, so i j p m i the complement of all the D j

S Gr n

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S

i by gluing T j i i to the trivial torsor T 0 j V i via t j along the isomorphism t on U j) j 2p i; T i; t i \V i = X i

i n i

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g of X S

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We construct the factorization maps along the diagonals. If we have coordinates x , ..., x = = x n, then we may set x ij = y i for all j2p i, where

Then S x j) = S i y i); since X S is the same in both cases, the

again single out a partition, but this time, none of the m

are disjoint, for i= 1;:::;n; denote by U

= S i = X n D i. n(~x; a t 1 i

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are an open cover of X

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) is in Gr

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nd we may de ne a G-torsor T has a natural trivialization t) T on X n U and the triple (X, S) Conversely given such a collection let T be the torsor obtained by gluing T, where on U over the open cover U, U is identified with T, which obviously satisfy the cocycle condition on triple intersec tions. Then T has a natural trivialization of TU ≅ X(S×) coming from the t and (x; T; t) ∈ G(S). For the S-equivariance, it is clear that at a point (x; T), both and are independent of the order of the coordinates of x.

, there is a description of Gr as a quotient of some \loop group" by some \arc group", bo th of them now in factorizable forms. Namely, they are

G(b (~x;g) ~x2X ~x2Xn(S);g2G xi) o

O)G((b X(S);g2G S;xb X) oS;xn [

b K)

where $\widehat{X}_{S;x}$ Then $Gr_n = G(\widehat{K})_n / G(\widehat{O})_n$ $X_{S;x}$ = Spec X_S), $b(\widehat{O}_{X_S}$, and thus both groups act on $Gr_{;D_n}$):i; this follows, as for the a ne grassmannia from the Beauville{Laszlo theorem. This is so similar to Grthat one is entitled to ask what the relation is, and the answer is simply that Gr_1 is a Gr_GG-bundle over X, where the structure group is the group b O).xIndeed, if we choose on some Zariski-open subset Uof Xa regular function zwhich is a local parameter at every point, then zidenti es each Owith b O= C [[z]] and thus identi es G(and G(b O) U thus their quotient with GrGb $O)_1$and G(b $K)_1$with G(b K) U U. The transition maps are obviously giv by elements of Aut(sometimes, a small complex disk (in the analytic topology), then Gr_1 = Grb O). T a useful conceptual notion, but its most practical form is that if Xis, as we will take it X. The relative lo and arc groups G(b $O)_n$and G(b $K)_{nG}$are factorizable in the same way as Gr_n(as made precise in the appendix).

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Convolution and the geometric Satake equivalence

Now we introduce the main object of study: the Hecke category. De nition 1. The n'th big Hecke category, denoted Sph_n, is the category of spherical, or G(b $O)_n$-equivariant D-modules on Gr_n; the regular Hecke category Sph is the category of G(b O)-equivariant D-modules on Gr. We will generally talk just about $Sph_{1G}$and Sph, and in the end we will state (without proof) the appropriate generalizations to Sph_n. The most important property of these categories is that they have convolution products, which are obtained by certain convolution diagrams. The most natural way of de ning convolution is to do it on G(b K) (or, indeed, G(b $K)_n$), via the multiplication map

G(b b

K) G(K)

A = $m_!$(A B

These maps in fact express Conv 1 2 = pr 1 e F_2 f(h)g(h_1 G 2) (g) R G

e F_2

= m

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1 b K) !GrG G 2Sph is F_1 Con vG (F1) q (F2

G (b O) G m ! G r G G ! G r 1;F2

F i

For any complexes A ;B

F of D-modules on G(b K), the formula B) is the geometric analogue of convolution of functio
Unfortunately, this de nition is not amenable to analysis since G(b K) is so wild. But suppos
sheaves F2Sph, and denote q: G(; then q) can be computed on a much better space. Inde
qcommutative) and thus(3), along with the objects on it, descends to the diagram:are G(b C
both the left and the right (which are di erent since Gis not, in general,= G(b K) Gr: (4) Co
\convolution diagram". There is one projection pr: Conv; it and mare de ned by the formulas
de nitions(2)and(1)):

as the product Grpr(g;(T;t)) = gmod G(b O) m(g;(T;t)) = (T;g t): Gr, but we will not want this
Rather, for F2Sph, we de nee Fto be the descent of prF) from the left-hand side of(3)to Con
F2Sph, Fe F 1: (5) Then the convolution of F
): (6)

Note that, a priori, this is merely a complex of D-modules and, indeed, makes sense for any equivariant to a map from the convolution di
complexes in the derived category. Later, we will show that it indeed sends Sph Sph to Sph.The program
established above is easily generalized to $Gr_1$and to the Grin general. Using the same words, the product
on $G(b K)_n$, $G(b K)_n$ X_n m_n $G(b K)!G(b K)_{nn}$(3) descends to the double quotient by actions of G(
b $O)_{n_n}$ n m n 0 0)

and admits, as before, one projection pr: $Conv_n$ $!Gr_n$
2 e Fon $Conv_n$, and we (F e n 0
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n
1 G(b K)$_n$ G (b O) Gr !Gr : (4
The left-hand side is denoted . When n= 1, this map is naturally identi ed with that of(4)over every point of X. For F2Sph(or indeed, a
$Conv_n$ equivariant complex), there is a twisted pullback
) As for(6), these are merely complexes of D-modules for now; we will return later to the question of how
these convolutions are related to that of(6). Returning to the ordinary grassmannian Gr G

G) as tensor categories, where
Once the equivalence Sph = Rep(
: are factorizable, the categories Sph_n
n Sp $!Sph_n$
h_m
e F∓ (pr F_1 e)[n] F F = m 1 2): (5
Just as the Gr F
, the theorem which is the
subject of these lectures is the geometric Satake equivalence:

Theorem 2. The convolution admits a commutativity constraint making Sph into a rigid tensor (\Tannakian") category. There
exists a faithful, exact tensor functor Sph !Vect inducing an equivalence (modulo a sign in the commutativity constraint) of
Sph with Rep(Gis the Langlands dual group of the reductive group G, whose weights are the coweights of Gand vice versa.
G) is established as categories, the convolution becomes less important, and is replaced by another form of factorizability
related to convolution on the Sph. We will digress from the proof in order to formulate a generalization of the above
theorem.
on them have a factorizable structure as well. Imprecisely, this
structure consists of the following data:
For any partition pof [1;n] into mparts, there is a direct image functor
! along p with Gr_m
of D-modules on U

p
Sph: Here the rst map is restriction from X_{nn_2}

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). Let q be the complete

corresponding to the identi cation of the restriction of Gr . This functor is right-exact and in fact has a right adjoint +n2in the derived cat

simplicity) that pis the partition n= nand that Uis the corresponding open set. O

category Sphpwhich are equivariant with respect to the action of (G(b O)n1 G(b

Sphn!Sphpp Sphn1

n to Up. For Fi p2Sph i on Fj p

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convolution in Sphn. The factorizable
structure of the Sphn
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, their image under the second map is (F F2)jUp. As

before, these maps admit right adjoints and, when n= 2, are actually exact. There is a version of the above point for ner partitions, and both of these maps are compatible with

corresponds to a factorizable notion ofG-representation. To separate the notion fromG, let Hbe any group. If F2D-mod(X), then we say that Hacts factorizably on Fif for every partition pof ninto mparts, there is an action of H, and these actions are consistent with re nement of p. This consistency is exempli ed by the following situation: let n= 3, and say that pis the partition 3 = 2 + 1 (in that order); then U= X

partition 3 = 1 + 1 + 1, so that U_q ^{n} is the complement of all the diagonals. On U_q, H
p, H_2 q, the rst factor of H_2
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The fusion pro duct

n(
0 Gr_G 0) is in fact a generalization of that (6) on Gr G = Gr_G
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The convolution
product (6
= j (Gr_1 F_1 F_2 Gr_1) by factorization. Then= $j_!$ j (F_0 1 F_0
2)[1] x[1]:

Lemma 4. Let F_i
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bX

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sw (F_0 1
where sw: X_2

!

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acts on F, and on Uacts on F; we require that restricted to Uact as the diagonal of the rst two factors of H, while the last factors act identically. We will denote by Rep(H) the category of such factorizable representations of Hin D-mod(X). The categories Rep(H) have the same factorizable structure as the Sph: a direct image along diagonals, and restriction and product maps away from the diagonals. Finally, we can state the big Satake equivalence:

Theorem 3. There are equivalences of categories identifying all the Sph with the Rep L
G);

this equivalence respects their factorizable structures as well as convolution.
We will only prove Theorem 2; Theorem 3 follows in a totally formal manner from it.
, at least as long as the objects being convolved are D-modules rather than complexes. The connection is via a local computation on X: suppose that X is a small complex disk with center denoted x, so that Gr= pr X. For F2Sph, let FF[1]
be its extension, along this product, to Gr . (It should be noted that the product decomposition of Gr
2 1 n !X2
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s canonical only up to the action of Aut(b O). However, it can be shown, in a manner not depending on the fusion product, that any element of Sph has a unique structure of Aut(b O)-equivariance, so that in fact this does not interfere with the arguments.) In this section, we will show that convolution on Sph has values again in Sph and that it has a natural commutativity constraint. The key is the following claim, which establishes convolution in Sph as a fusion product, so called because convolution at a point x2X is obtained via tensor product over two points y;z2X which come together (or \fuse") at x.
G their extensions as above to Gr . Let j: X2
r is that if : X!X
!
1 0 1 2pr 0 2) = pr 0 1 pr 0 1 = pr 1 0 2pr 0 1 = F F
F
F2 1 2 b nx
!X X
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0
S;x
2
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be
the inclusion, and identify j
Note that this product depends only on the factorization structure of Gr. One of the properties of j! is the inclusion of the diagonal, then (M)[1] is a D-module for any D-module M(rather than, as it is a priori, a complex on X). This immediately implies that Fis a D-module. It also gives a commutativity constraint for , coming from the isomorphism

F) = sw (pr F F F F F F $^{0}_{2}$ $^{0}_{1}$

2 2 1 1 2

swaps the coordinates and, of course, sw = , so the above isomorphism indeed gives an isomorphism of Fwith F F_1.Lemma 4shows why it is necessary to work in the abelian category Sph, rather than the derived category in which the de nitions of convolution also make sense: the operation jis only a functor on D-modules. Thus, we need only proveLemma 4. In order to set up the core theoretical argument, we introduce the convolution grassmannian f Gr. Once again, we give a quick (though correct) de nition here and defer a technical development to the appendix. Recalling(4), let f Grbe the closed subscheme of pairs ((~x;g);(~x;T;t)) in Convwith the following properties: As an element of Gn(x) [x)) , gextends to$_{S;x}$);

The trivialization t, de ned on X_S b n(x_1) [x_2)), extends to X_S nx$_2$ with Gr$_1$

O
)

Xin Gr$_2$. Using them, we construct a twisted product F_1

0 i= pr Gr$_1$in the following way: Let F
i[1] on Gr$_1$ Xand X Gr respectively, considered as objects of Sph ;
f Gr$_2$j F = Conv$_1$. Furthermore, the map mof(4,!Gr$_2$ 0)induces a map, likewise called m, from f Gr to Gr
). Both of these conditions are invariant under multiplication by G(2, so do in fact de ne a subfunctor. It is evident from this de nition that over Xn , there is a natural identi cation of f Gr, and that. There is a \cheap" inclusion X Gr, sending a pair (x;(y;T;t)) to ((x;y);T;t); likewise, there is an inclusion of Gr
The tensor product (q -biequivariant and so descends to Conv$_2$; The descended D-module F_1e F_{21}happens to live on
De nition 5. The outer convolution of F;$F_2$2Sph$_1$is F_{12} oF_2= m (F_{11}e F). Clearly, F F_2= (F_1 oF_{22})[
proveLemma 4, it su ces to prove (going back to $F_{0\,1}$2Sph) F o0 2F= j! j ($F_{0\,1}$ $F_{0\,2i}$): (7) To do this, we introduce a catalyst in the form of the unipotent nearby and vanishing cycles functors; rather than giving a detailed discussion of them, we refer the reader to the notes [2] on Beilinson's paper [1]. Here, only the following properties are important (once again, the D-modules are holonomic):

For any scheme Y and Cartier divisor D Y with open complement U, there is a functor of unipotent nearby cycles around D, : D-mod(U) !D-mod(D), together with an endomorphism (unipotent on each un Dun D(F)) called the monodromy. There is likewise a functor un D: D-mod(Y) !D-mod(D) of unipotent vanishing cycles.

Let j: U!Y be the inclusion. Suppose that F2D-mod(Y) and that un D(j = j! (j F) has trivial monodromy; then a necessary and su cient condition that FF) is that un D Y-module, then it has both of these properties. When this happens, then i (F) = 0. If Fis a free OF[1] = i!F[1] = un D(F), where iis the inclusion of D. (This is the only one of these facts that relies on the theory from Beilinson's paper.)
is local on Din that for any open set V and F2D-mod(U), we have un D(F)jVun D = un D(FjV

un 2D-mod(Y), FF G), F0 i2D-mod(F). Then we have un E(FY FF) = un D(FY) FF F
Y 1;F2 1 un E = un D 0 1

1

1
un D), and this isomorphism respects the monodromy. This is likewise true for 1 Yand F2D-mod(Y). Nearby and vanishing cycles respect products, as follows: let Z= Y F, set E= pr(D), and let Fand likewise for , and this isomorphism respects the monodromy. If p: Z!Y is a proper morphism and E= p(D), then p p(nearby cycles commute with proper direct image) and this isomorphism respects the monodromy. Likewise, vanishing cycles commute with proper direct image.
The glue that makes this all stick together is the following easy lemma: Lemma 6. . Then F 0 2
Let F2D-mod(Gr2D-mod(Gr) their extension to Gr
0 = GrG = pr (F_1 F_2)[2]:
1 1 7

$F_{0\,2}$
has no vanishing cycles and its nearby cycles have trivial monodromy.
Proof. We continue to identify Gr$_1$ X, and we write pr to mean (in this proof) the projection Gr Gr$_1$!(GrG)$_2$. Then we have F

in the statement that the cycles functors respect products, and let F_Y
F
1 0 0 1 G = Gr Gr) that f
0 2 1 S Gr

un
0 1

Take Y = X^2, D= , and F= Gr_2 $_2$ with the trivial D-module structure. Then it has no vanishing cycles or nearby-cycles m
be O_X the same is true of the tensor product (which, to be precise, we take to be F). The proo
chaining together the above properties. It turns out (one can argue directly,
or see the appendix; either way, this is analogous to the fact that Conv G G 2
0 1($F_{0 1}$ o F Gr_1. Thus, Lemma 6 applies, so) is trivial. u e F_0 away from .
Since m is a proper map, m n 2
0 2 2
(
F
o F (F 0 2) = j (F F) by the factorizability of f Gr
is locally
isomorphic to Gr) = 0 and the monodromy action on e $F_{0 2}$ preserves these properties, so the same is true
of F, and the criterion for it to equal the minimal extension of its own restriction applies. To complete the proof,
we note that j

App endix: the convolution grassmannian

In this appendix, we discuss the convolution grassmannian more formally. There are in fact many variations,
but we only need one:
[1]f (S) = ;x_2;T_1;T 1) is an isomorphism T = T_2 on $X_S$nx nx on X_S t is a trivialization of T_1 2;t;) 2
Gr
The reason for its existence is that it admits the diagram (4 0)
:
8 x_i 2X(S);T_i are G-torsors on X 9
>< >=
>: 2 1 >;
(x :

Gr_1 pr 2 m !Gr :
2, though: just set
resembles a product of Gr_1^2 2 f Gr 1 2 ; t):
2);T
2
Clearly, f [1]with itself, but that product does not admit a map such as m. The existence of r
Gr de nition off Gr
m(x_1;x ;T ;T ;t;) = ((x ;x 2
Likewise, pr sends such a point to (x_1;T_1;t). Just like the Gr_n, f Gr_2 is ind-proper, hence m is a proper map.

Although it is not actually the product Gr_1 Gr_1, the projection map pr is in fact a Gr-bundle over Gr_1. To see this, we de ne the following functor:1

e

G(b O) It is easy to see that Gr_1^1(S) = (G(b O)1 (x_1;

$(x_1;T;t_1) \in Gr(S)$ t_{21} is a trivialization of T on b X

1 ; 3 . The restriction of t_2 to b 2 nx

T 3 0; let T_2 X

), and so

2

2 away from x

$(x_1;x_2;T;t_1;t_2) \in G(b\,O)$ (note the equality of X-coordinates); let 0;t_3) $\in Gr_1(S)$ 2

$T_{11}(S)$ (x= T and t= t_{12}; T is a trivialization of T the restriction of T_0 to b S;x

$X_{S;x_2}$ along t t_1 2 has an isomorphism with T_{12}

$(x_1;x_2$ 1;T_2;t;) $\in$ f Gr (S):

8

) is a trivialization of T, and the like restriction of t be the G-torsor obtained by gluing to T, using the Beauville{Laszlo theorem. Then by de nition, T

21This gives the map(8). To see that it is surjective, take a point such as the one above and let0be T2 2 1
T) [x2), thus obtaining a trivialization t3;T0;t3) 2Gr1(S). As before, we take T= T1b
XS;x22(arbitrarily) on each set Uof this cover and take t1= t; then (x1;x2;T;t1;t2
this we have already identi ed each ber with G(b X), as desired. Let be the projection onto
Gr1U;x2from the left-hand side of(8). If F2Sph1, then e Fonf Gr2; as before, for
F1;F22Sph11e F2= pr F1 e F2.

App endix: factorizable structure

In this appendix, we give a rigorous description of the factorizable structure on the Grn
1: I1 !J1 and p2: I2 !J2 to their union p: I1 [I2 !J1 = p1(j) for the j'th part of this partition. We de ne two k
p!J, a rst re nement is a partition r1: I!I0such that p= p
j a partition r2: J00 r!Jsuch that p= r120 p0
[J2
away from xglued, via t, to the trivial torsor away from x) and a point (x, but it is not necessarily possible to
trivialize it on. However, since Tis a torsor, there is an open cover of Son which such trivializations exist, and
we pick one t) 2 e G(b O)1(U). Thus,(8)is surjective as a map of Zariski sheaves (let alone fppf sheaves).
Finally, in the course of showing
(F) is G(b O)equivariant and therefore descends to a D-module, we de ne the twisted product F

. This requires some
abstract nonsense with partitions of nite sets; thus, we introduce the additional notation: for any nite sets
Iand J(thought of as \index sets"), a partition of Iinto Jparts is a surjection p: I!J. We will write p

. Note the directions of
the maps. Let Part be the category of partitions whose morphisms are generated
by the re nements of both types. There is a natural bifunctor Un: Part Part !Part sending a pair of partitions
p; this functor admits a natural commutativity constraint.
Let Xbe a scheme (it may as well be our curve). For an index set I, let XI = Qi 2I
I j r: U to the coordinates (x j : p 1 = x 2, then p(i) = p(i
p I 2) such that if x 0
U 0 I 0
p: Up
p
p

p
p
p: XJ !XI sending x i ji2p 0 i), with image i i 1 2
0into U
p0!U !J
= ir1 jp
2 1 is a rst re nement, let I
= j r2
is a second re nement, let I , we have U
0)
Xbe the unordered power of Xcorresponding to this nite set. For any partition p: I!J, there is an induced
closed immersion i. There is also a corresponding open subset Uof X(not its complement) consisting of all

points (x); let j!Xbe the open immersion. For any partition pand morphism r: p!pin Part, there is a locally closed immersion l_{rp_0}, which clearly sends Ude ned as follows for the re nements: If r= r_{rp}; If r= r, which again clearly has image in U. One should check that for any p;$p_{0Un(p;pp}$ U_{p_0}. Let PSch (\schemes over partitions") be the category, bered over Part, such that for any partition p,
the ber PSch is Sch=U_p, the category of schemes over U_p

$_1$=U ^{1} and X =U_{p_2} r 0 to (X_1 X_2)j$U_{Un(p_1;p_2)}$ p

, and let the cartesian morphisms (pullbacks along morphisms r) be given by restriction along l. There is again a bifunctor Pr: PSch PSch !PSch sending X, admitting a natural commutativity constraint. If : PSch !Part is the structure functor, then identi es Pr with Un. In more usual terms, the two categories are braided monoidal categories and is a braided monoidal functor.

De nition 7. An sf-scheme (\symmetric factorizable scheme") is a braided monoidal section functor Fof . This means:

9 F(p) with F(p

1.We have F= id exactly (not up to isomorphism); 2.For $_0$, and these isomorphisms are functorial in r; every morphism r: p!p, there is an isomorphism of r

3.There is the additional datum of an isomorphism of functors making the square commute:
PrPSch PSch PSch

UnPart Part Part

4.This isomorphism is required to be compatible with the commutativity constraints in the sense that if Sw is the functor swapping factors in either product category of the above diagram, then the following diagram of functors and natural transformations commutes:

Pr Sw Pr

Un Sw Un

If for every index set I, having cardinality #I= n, we have F(I!f1g) = Gr_n, then Fis a factorizable structure on $Gr_{nG;X}$, and in the main text we have described one such structure. The correspondence between the above properties and the ones given before is:

The existence of factorization along diagonals (the rst factorization property) is a special case of (2) when r= $r_{1p}$is a rst re nement and pis the trivial partition I !f1gwith only one part (so U_p= X_I). Factorization on diagonal complements (the second factorization property) is a combination of (3) andthe special case of (2) with r= r_{n2}-equivariance is special case of (2) in which pis the trivial partition and p_0a second re nement and pthe trivial partition. The S= p, so that ris an automorphism of I.

Compatibility of the three structures above is the stipulation in (2) that the isomorphisms be functorial, together with the functoriality of Pr and the fact that is a monoidal functor. The role of (4) is to ensure that the data of S_n-equivariance on $Gr_n$is compatible with the natural S-equivariance of a product Gr_{n_1} $Gr_{n_2}$when both are identi ed on U_p(here pis the partition n= n_{21}+ n_2).

References

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