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**ATLANTIS STUDIES IN PROBABILITY AND STATISTICS**

VOLUME 2

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# Atlantis Studies in Probability and Statistics

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# Stochastic Differential Games

*Theory and Applications*

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*Dedications to our families:*

*Usha, Vikas, Vilas and Varsha Ramachandran*

*and*

*Debbie, Mathew, Jonathan, and Maria Tsokos*

# Preface

Conflicts in the form of wars, or competition among countries and industrial institutions are plenty in human history. The introduction of game theory in the middle of the twentieth century shed insights and enabled researchers to analyze this subject with mathematical rigor. From the ground-breaking work of VonNeumann and Morgenstern, modern game theory evolved enormously. In the last few decades, Dynamic game theory framework has been deepened and generalized from the pioneering work on differential games by R. Isaacs, L.S. Pontryagin and his school, and on stochastic games by Shapley. This book will expose the reader to some of the fundamental methodology in non-cooperative game theory, and highlight some numerical methods, along with some relevant applications.

Since the early development days, differential game theory has had a significant impact in such diverse disciplines as applied mathematics, economics, systems theory, engineering, operations, research, biology, ecology, environmental sciences, among others. Modern game theory now relies on wide ranging mathematical and computational methods, and relevant applications that are rich and challenging. Game theory has been widely recognized as an important tool in many fields. Importance of game theory to economics is illustrated by the fact that numerous game theorists, such as John Forbes Nash, Jr., Robert J. Aumann and Thomas C. Schelling, have won the Nobel Memorial Prize in Economics Sciences. Simply put, game-theory has the potential to reshape the analysis of human interaction.

In Chapter 1, we will present a general introduction, survey, and background material for stochastic differential games. A brief introduction of Linear pursuit-Evasion differential games will be given in Chapter 2 for a better understanding of the subject concepts. Chapter 3 will deal with two person Zero-sum stochastic differential games and various solution methods. We will also introduce games with multiple modes. Formal solutions for some classes of stochastic linear pursuit-evasion games will be given in Chapter 4. In Chapter 5, we will discuss  $N$ -person stochastic differential games. Diffusion models are in general

not very good approximations for real world problems. In order to deal with those issues, we will introduce weak convergence methods for two person to the stochastic differential games in Chapter 6. In Chapter 7, will cover weak convergence methods for many player games. In Chapter 8, we will introduce some useful numerical methods for two different payoff structure; discounted payoff and ergodic payoff as well as the case of nonzero sum games. We will conclude the book in Chapter 9 by giving some real world applications of stochastic differential games to finance and competitive advertising.

We wish to express our sincere appreciation to the reviewers of the preliminary manuscript of the book for their excellent comments and suggestions.

Dr. M. Sambandham, Professor of Mathematics, Chief Editor, International Journal of Systems and applications.

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K.M. Ramachandran

C.P. Tsokos

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