A Probability Metrics Approach to Financial Risk Measures

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To my grandchildren Iliana, Zoya, and Zari

SVS

To my parents Veselin and Evgeniya Kolevi and
my brother Pavel Stoyanov

FJF

To my wife Donna and
my children Francesco, Patricia, and Karly

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Preface

The theory of probability metrics is a branch of probability theory. It finds application in different theoretical and applied fields such as probability theory, queuing theory, insurance risk theory, and finance. The theory of probability metrics looks for answers to the following basic question: How can one measure the difference between random quantities? In finance, for example, we assume a stochastic model for asset return distributions and, in order to estimate the risk of a portfolio of assets, we sample from the fitted distribution. Then, we use the generated simulations to calculate portfolio risk. In this context, there are two issues arising on two different levels. First, the assumed stochastic model should be "close" to the empirical data. In this sense, we say that we need a realistic model in the first place. Second, since the risk estimate is essentially computed from random scenarios, we have to be aware of the variability of the estimator and how it depends on the assumed asset return distributions.

Although based on universal principles and ideas, the field of probability metrics is very specialized. Most of the literature is highly technical and is accessible mostly to specialists in probability theory. As far as applications are concerned, apart from our book *Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: Ideal Risk, Uncertainty, and Performance Measures* (John Wiley & Sons, 2008), we are unaware of other literature describing applications in finance.

PREFACE

This book has two goals. The first goal is to describe applications in finance and extend them where possible. The second goal is to present the theory of probability metrics in a more accessible form which would be appropriate for non-specialists in the field. Topics requiring more mathematical rigor and detail are included in technical appendices to chapters.

The book is organized in the following way. Chapter 1 provides a conceptual description of the method of probability metrics and reviews direct and indirect applications in the field of finance. Chapter 2 provides an introduction to the theory of probability metrics. The classical theory describing investor choice under uncertainty is provided in Chapter 3. Chapter 4 discusses the classification of probability distances to primary, simple, and compound types. The information in Chapter 2 is a prerequisite. Chapters 5, 6, and 7 are devoted to risk and uncertainty measures and discuss in detail AVaR and the Monte Carlo method for AVaR estimation. Chapter 6 is a prerequisite to Chapter 7. Finally, Chapter 8 considers the problem of quantifying stochastic dominance relations and takes advantage of the terms introduced in Chapter 3.

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About the Authors

Svetlozar (Zari) T. Rachev completed his Ph.D. degree in 1979 from Moscow State (Lomonosov) University, and his Doctor of Science Degree in 1986 from Steklov Mathematical Institute in Moscow. Currently he is Chair-Professor in Statistics, Econometrics and Mathematical Finance at the University of Karlsruhe in the School of Economics and Business Engineering. He is also Professor Emeritus at the University of California, Santa Barbara in the Department of Statistics and Applied Probability. He has published seven monographs, eight handbooks and special-edited volumes, and over 300 research articles. His recently coauthored books published by Wiley in mathematical finance and financial econometrics include *Fat-Tailed and Skewed Asset Return Distributions: Implications for Risk Management, Portfolio selection, and Option Pricing* (2005), *Operational Risk: A Guide to Basel II Capital Requirements, Models, and Analysis* (2007), *Financial Econometrics: From Basics to Advanced Modeling Techniques* (2007), and *Bayesian Methods in Finance* (2008). Professor Rachev is cofounder of Bravo Risk Management Group, specializing in financial risk-management software. Bravo Group was recently acquired by FinAnalytica, for which he currently serves as Chief-Scientist.

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Frank J. Fabozzi is Professor in the Practice of Finance in the School of Management at Yale University. Prior to joining the Yale faculty, he was a Visiting Professor of Finance in the Sloan School of Management at MIT. Professor Fabozzi is a Fellow of the International Center for Finance at Yale University and on the Advisory Council for the Department of Operations Research and Financial Engineering at Princeton University. He is the editor of the *Journal of Portfolio Management*. His recently co-authored books published by Wiley in mathematical finance and financial econometrics include *The Mathematics of Financial Modeling and Investment Management* (2004), *Financial Modeling of the Equity Market: From CAPM to Cointegration* (2006), *Robust Portfolio Optimization and Management* (2007), *Financial Econometrics: From Basics to Advanced Modeling Techniques* (2007), and *Bayesian Methods in Finance* (2008). He earned a doctorate in economics from the City University of New York in 1972. In 2002 Professor Fabozzi was inducted into the Fixed Income Analysts Society's Hall of Fame and he is the 2007 recipient of the C. Stewart Sheppard Award given by the CFA Institute. He earned the designation of Chartered Financial Analyst and Certified Public Accountant.